NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Extension 2

Assessment Task 2 Term 2 2008

Name:_______________________________ Mathematics Class:_________

Time Allowed: 70 minutes + 2 minutes reading time

Available Marks: 53

Instructions:

- Questions are *not* of equal value.
- Start each question on a new page.
- Put your name on the top of each page.
- Attempt all three questions.
- Show all necessary working.
- Marks may be deducted for incomplete or poorly arranged work.
- Write on one side of the page only. Do not work in columns.
- Each question will be collected separately.
- If you do not attempt a question, submit a blank page with your name and the question number clearly displayed.

Question	1	2	3	Total	
E3		/20		/20	
E4	/19			/19	
E9			/14	/14	
				/53	%

Question 1: (19 marks) **Marks**

a) i) Prove that if α is a double root of the polynomial equation $P(x) = 0$, then $P(\alpha) = P'(\alpha) = 0$. **2**

ii) $P(x) = 12x^3 - 8x^2 - x + 1$ has a double root. Factorise $P(x)$ into irreducible factors over the real numbers. **3**

b) i) If $1 + i$ is a zero of the polynomial $f(x) = 4x^3 - 7x^2 + 6x + 2$ explain why $1 - i$ is also a zero. **1**

ii) Hence factorise $f(x)$ over the complex numbers. **2**

c) The equation $x^3 + 2x - 1 = 0$ has roots α, β and γ.

i) Find an equation with roots $\alpha^2, \beta^2, \gamma^2$. **3**

ii) Hence evaluate $\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2$. **1**

iii) Find an equation with roots $\alpha, -\alpha, \beta, -\beta, \gamma, -\gamma$. **2**

d) i) Decompose $\dfrac{x^2 + x + 2}{(x+2)(x^2+4)}$ into partial fractions over the real field. **3**

ii) Hence show that $\displaystyle\int_0^2 \frac{x^2 + x + 2}{(x+2)(x^2+4)}\,dx = \frac{3}{4}\log 2$. **2**

Question 2: (20 marks) **Start a new page** **Marks**

a) Consider the hyperbola $x^2 - 4y^2 = 4$.

i) Find α) the eccentricity; **1**

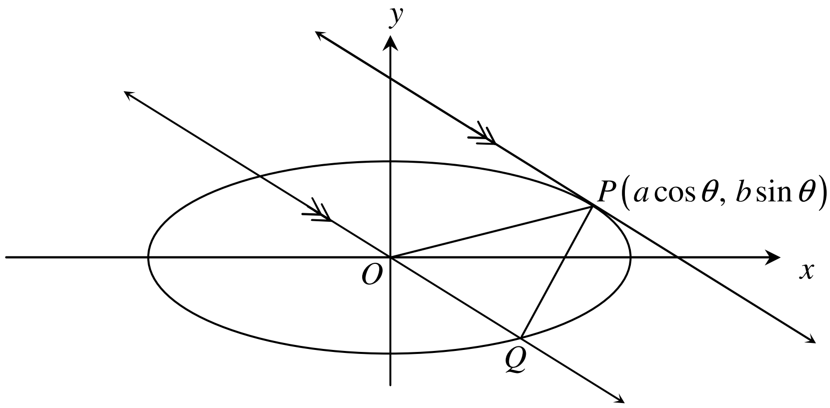
β) the co-ordinates of the foci; **1**

γ) the equations of the directrices; **1**

ii) Hence sketch the curve indicating the asymptotes and these details. **1**

b) In the diagram, $P\left(a\cos\theta,\, b\sin\theta\right)$ is a point on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where P is in the first quadrant.

A straight line through the origin, parallel to the tangent at P meets the ellipse at the point Q, where both P and Q lie on the same side of the y-axis.



i) Prove that the equation of the line OQ is $bx\cos\theta+ay\sin\theta=0$. **3**

ii) Show that the coordinates of Q are $\left(a\sin\theta,-b\cos\theta\right)$. **3**

iii) By considering the distance of P from OQ or otherwise, prove that the area of the triangle ΔOPQ is independent of the position of P. **3**

b) $P\left(cp,\, \dfrac{c}{p}\right)$ and $Q\left(cq,\, \dfrac{c}{q}\right)$ where $p > 0$ and $q > 0$, are points on the rectangular hyperbola $xy = c^2$. The tangents to the rectangular hyperbola at P and Q intersect at the point R.

You are given that the tangent to the rectangular hyperbola at the point $P\left(cp,\, \dfrac{c}{p}\right)$ has the equation $x + p^2y = 2cp$. Do not prove this result.

i) Show that the coordinates of R are $\left(\dfrac{2cpq}{p+q},\dfrac{2c}{p+q}\right)$. **2**

ii) P and Q are variable points on the rectangular hyperbola which move so that PQ always passes through $(2c, c)$. Show that $p + q = 2 + pq$. **3**

iii) Find the equation of the locus of R. **2**

Question 3: (14 marks) **Start a new page** **Marks**

a) Let $P(z) = z^7 - 1$.

i) Use de Moivre's theorem to write down the roots of $P(z) = 0$ and illustrate their relationship on an Argand diagram. **2**

ii) Hence write $P(z)$ as a product of real linear and quadratic factors. **2**

iii) Write down the quotient when $P(z)$ is divided by $z - 1$ in two different forms. **2**

iv) Hence show that $\left(1-\cos\frac{2\pi}{7}\right)\left(1-\cos\frac{4\pi}{7}\right)\left(1-\cos\frac{6\pi}{7}\right) = \frac{7}{8}$. **2**

b) In order to recognise anagrams quickly, Miss V always arranges the available letters in a circle.

She is currently considering the letters of the word 'SURRENDER'.

i) In how many different ways can all of the letters be arranged in a circle? **1**

ii) If she forms one such circular arrangement at random, what is the probability that the word 'NERDS' occurs without a break in a clockwise direction within the arrangement? **3**

iii) Now she selects 5 letters at random. What is the probability that the letters selected are the letters of the word 'NERDS'? **2**

End of Paper

Question 1:

a) i) Let α be a double root of $P(x) = 0$, then $P(\alpha) = 0$.

Also $P(x) = (x - \alpha)^2Q(x)$ for some polynomial $Q(x)$

Now $P'(\alpha) = (x - \alpha)^2Q'(x) + 2(x - \alpha)Q(x)$

$\therefore P'(\alpha) = (\alpha - \alpha)^2Q'(\alpha) + 2(\alpha - \alpha)Q(\alpha)$

$= 0$

$\therefore P(\alpha) = P'(\alpha) = 0$ as required

ii) $P(x) = 12x^3 - 8x^2 - x + 1$

$P'(x) = 36x^2 - 16x - 1$

$= (18x + 1)(2x - 1)$

$\therefore$ possible double roots are $-\frac{1}{18}$ or $\frac{1}{2}$

But 18 is not a factor of 12 so the double root must be $\frac{1}{2}$

$\therefore P(x) = (2x - 1)^2(3x + 1)$

b) i) $f(x) = 4x^3 - 7x^2 + 6x + 2$ has real coefficients and so complex roots will occur in conjugate pairs

$\therefore$ as $1 + i$ is a root $1 - i$ is also a root.

ii) Now $(x - 1 - i)(x - 1 + i) = (x - 1)^2 - i^2$

$= x^2 - 2x + 2$

$\therefore f(x) = 4x^3 - 7x^2 + 6x + 2$

$= (x^2 - 2x + 2)(4x + 1)$

$= (x - 1 - i)(x - 1 + i)(4x + 1)$

c) Let $P(x) = x^3 + 2x - 1 = 0$ have roots $x = \alpha, \beta$ and γ.

i) For an equation with roots $y = \alpha^2, \beta^2, \gamma^2$ the relationship is $y = x^2$

Then $x = \sqrt{y}$

$\therefore P(x) = 0$

becomes $P\left(\sqrt{y}\right) = 0$

$$\left(\sqrt{y}\right)^3 + 2\left(\sqrt{y}\right) - 1 = 0$$

$$y\sqrt{y} + 2\sqrt{y} + 1 = 0$$

$$\sqrt{y}\left(y + 2\right) = -1$$

$$\sqrt{y} = \frac{-1}{y+2}$$

$$y = \frac{1}{y^2 + 4y + 4}$$

$$\therefore y^3 + 4y^2 + 4y - 1 = 0$$

ii) $\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2$ is the sum of the roots in pairs from i) above

$\therefore \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 = \frac{c}{a}$

$= 4$

iii) If $ax^3 + bx^2 + cx + d = 0$ has roots α, β and γ, an equation with roots $-\alpha, -\beta, -\gamma$ is $ax^3 - bx^2 + cx - d = 0$.

$\therefore$ $x^3 + 2x + 1 = 0$ has roots $-\alpha, -\beta, -\gamma$

and $(x^3 + 2x - 1)(x^3 + 2x + 1) = 0$ has roots has roots $\alpha, \beta, \gamma, -\alpha, -\beta, -\gamma$

$$(x^3 + 2x - 1)(x^3 + 2x + 1) = 0$$
$$(x^3 + 2x)^2 - 1 = 0$$
$$x^6 + 8x^4 + 4x^2 - 1 = 0$$

b) i)
$$\frac{x^2+x+2}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$$
$$x^2+x+2 = A(x^2+4) + (Bx+C)(x+2)$$

If $x = 2$: $\quad 4-2+2 = 8A$
$$A = \frac{1}{2}$$

If $x = 0$: $\quad 2 = 4A + 2C$
$$2 = 2 + 2C$$
$$C = 0$$

If $x = 1$: $\quad 1+1+2 = 5A + (B+C)3$
$$4 = \frac{5}{2} + 3B$$
$$3B = \frac{3}{2}$$
$$B = \frac{1}{2}$$

$$\therefore \quad \frac{x^2+x+2}{(x+2)(x^2+4)} = \frac{1}{2(x+2)} + \frac{x}{2(x^2+4)}$$

ii)
$$\int_0^2 \frac{x^2+x+2}{(x+2)(x^2+4)}\,dx = \int_0^2 \frac{1}{2(x+2)}\,dx + \int_0^2 \frac{x}{2(x^2+4)}\,dx$$
$$= \frac{1}{2}\int_0^2 \frac{1}{(x+2)}\,dx + \frac{1}{2}\cdot\frac{1}{2}\int_0^2 \frac{2x}{(x^2+4)}\,dx$$
$$= \left[\frac{1}{2}\ln(x+2) + \frac{1}{4}\ln(x^2+4)\right]_0^2$$
$$= \frac{1}{2}\ln 4 + \frac{1}{4}\ln 8 - \left(\frac{1}{2}\ln 2 + \frac{1}{4}\ln 4\right)$$
$$= \frac{1}{2}\ln 4 + \frac{1}{4}\ln 8 - \frac{1}{2}\ln 2 - \frac{1}{4}\ln 4$$
$$= \frac{1}{2}\ln 2^2 + \frac{1}{4}\ln 2^3 - \frac{1}{2}\ln 2 - \frac{1}{4}\ln 2^2$$

$$= \ln 2 + \frac{3}{4}\ln 2 - \frac{1}{2}\ln 2 - \frac{1}{2}\ln 2$$

$$= \frac{3}{4}\ln 2 \qquad \text{as required}$$

Question 2:

a) $x^2 - 4y^2 = 4$ becomes $\frac{x^2}{4} - y^2 = 1$.

i) α) $a = 2$ and $b = 1$

$$b^2 = a^2\left(e^2 - 1\right)$$

$$1 = 2^2\left(e^2 - 1\right)$$

$$\frac{1}{4} = e^2 - 1$$

$$e^2 = \frac{5}{4}$$

$$e = \frac{\sqrt{5}}{2} \qquad \text{as } e > 0$$

β) $ae = \sqrt{5}$ $\therefore$ foci are at $\left(\pm\sqrt{5}, 0\right)$

γ)

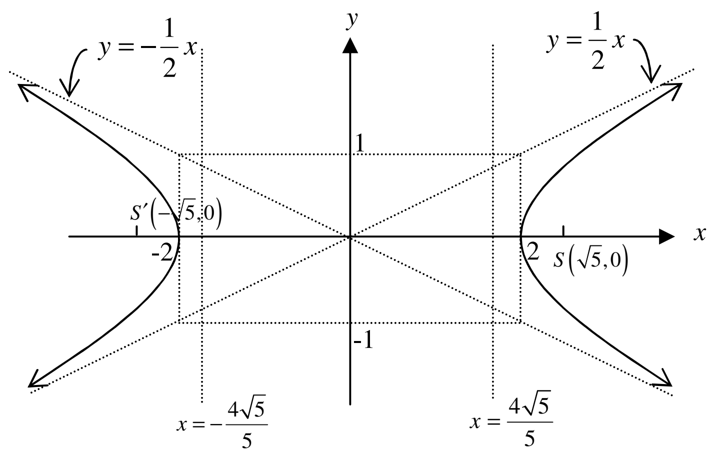
$$\frac{a}{e} = \frac{2}{\frac{\sqrt{5}}{2}}$$

$$= \frac{4}{\sqrt{5}}$$

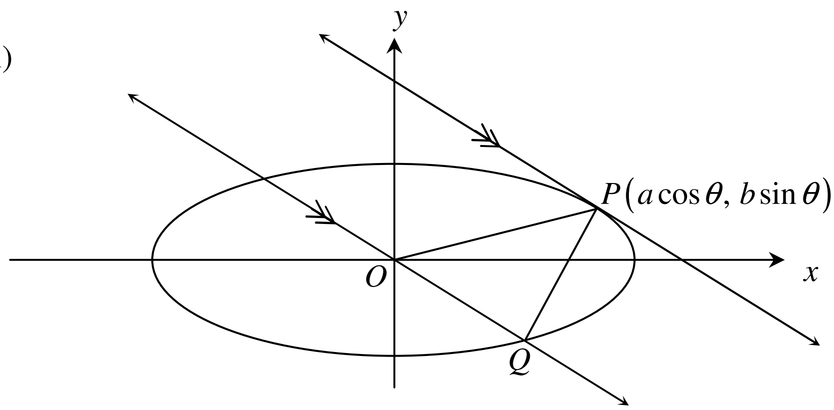
$$= \frac{4\sqrt{5}}{5}$$

$\therefore$ the equations of the directrices are $x = \pm\frac{4\sqrt{5}}{5}$

ii)



b) i)



At P: $x = a\cos\theta$ and $y = b\sin\theta$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$= b\cos\theta \times \frac{1}{-a\sin\theta}$$

$$= -\frac{b\cos\theta}{a\sin\theta}$$

$\therefore$ equation of the line OQ is

$$y = -\frac{b\cos\theta}{a\sin\theta}(x)$$

$$ay\sin\theta = -bx\cos\theta$$

$\therefore \quad bx\cos\theta + ay\sin\theta = 0$ as required

ii) For Q substitute $y = -\dfrac{b\cos\theta}{a\sin\theta}(x)$ into $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Then $\dfrac{x^2}{a^2} + \dfrac{b^2x^2\cos^2\theta}{b^2a^2\sin^2\theta} = 1$

$$x^2\left(1 + \frac{\cos^2\theta}{\sin^2\theta}\right) = a^2$$

$$x^2\left(\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}\right) = a^2$$

$$x^2 = a^2\sin^2\theta$$

$$x = \pm a\sin\theta \quad \text{but } x > 0$$

$$x = a\sin\theta$$

Substituting into OQ gives $y = -b\cos\theta$

$\therefore$ the coordinates of Q are $(a\sin\theta, -b\cos\theta)$

iii) The distance of P $(a\cos\theta, b\sin\theta)$ from OQ: $bx\cos\theta + ay\sin\theta = 0$ is

$$h = \frac{|b(a\cos\theta)\cos\theta + a(b\sin\theta)\sin\theta|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$= \frac{ab(\cos^2\theta + \sin^2\theta)}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \quad \text{as } a,\ b > 0$$

$$= \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

The distance OQ is $b=\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$

$\therefore\ A=\frac{1}{2}bh$

$$A=\frac{1}{2}\cdot\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}\cdot\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}$$

$=\frac{ab}{2}$ which is independent of θ and consequently the position of P

c) i) At P: $x+p^2y=2cp$ (1)

At Q: $x+q^2y=2cq$ (2)

(1) − (2): $\left(p^2-q^2\right)y=2c\left(p-q\right)$ but $p\neq q$

$\therefore$ $y=\frac{2c}{p+q}$

Substitute into (1): $x+p^2\left(\frac{2c}{p+q}\right)=2cp$

$$x=\frac{2cp\left(p+q\right)-2cp^2}{p+q}$$

$$=\frac{2cpq}{p+q}$$

$\therefore$ $R=\left(\frac{2cpq}{p+q},\frac{2c}{p+q}\right)$ as required

ii) $m_{PQ}=\frac{\frac{c}{p}-\frac{c}{q}}{cp-cq}$

$$=\frac{q-p}{pq\left(p-q\right)}$$

$$=-\frac{1}{pq}$$

$\therefore$ equation PQ: $y-\frac{c}{p}=-\frac{1}{pq}\left(x-cp\right)$

$pqy-cq=-x+cp$

$x+pqy=c\left(p+q\right)$

This passes through $\left(2c,c\right)$

$\therefore$ $2c+pqc=c\left(p+q\right)$

ie: $2+pq=p+q$ as required

iii) At R: $x=\frac{2cpq}{p+q}$ (1)

$y=\frac{2c}{p+q}$ (2)

Also $2+pq=p+q$ (3)

From (1) and (2): $x = pqy$

$\therefore \quad pq = \dfrac{x}{y}$

From (2): $p + q = \dfrac{2c}{y}$

Substituting both these into (3): $2 + \dfrac{x}{y} = \dfrac{2c}{y}$

$2y + x = 2c$

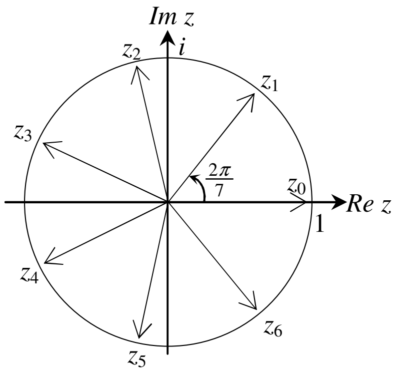
$\therefore$ the locus of R is: $x + 2y - 2c = 0$

Question 3:

a) i) $P(z) = z^7 - 1 = 0$

$z^7 = 1$

$z = \cos\left(\frac{2n\pi}{7}\right) + i\sin\left(\frac{2n\pi}{7}\right)$ for $n = 0,1,2,3,4,5,6$



ii) By symmetry, $z_1 = \overline{z}_6;\ z_2 = \overline{z}_5;\ z_3 = \overline{z}_4$

$\therefore\ (z - z_1)(z - z_6) = z^2 - 2z\,\mathrm{Re}\,z_1 + 1$

$(z - z_2)(z - z_5) = z^2 - 2z\,\mathrm{Re}\,z_2 + 1$

$(z - z_3)(z - z_4) = z^2 - 2z\,\mathrm{Re}\,z_3 + 1$

Also, $z_0 = 1$

$\therefore\ z^7 - 1 = (z - z_0)(z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5)(z - z_6)$

$= (z-1)\left(z^2 - 2z\,\mathrm{Re}\,z_1 + 1\right)\left(z^2 - 2z\,\mathrm{Re}\,z_2 + 1\right)\left(z^2 - 2z\,\mathrm{Re}\,z_3 + 1\right)$

$= (z-1)\left(z^2 - 2z\cos\frac{2\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{6\pi}{7} + 1\right)$

iii) $\dfrac{z^7 - 1}{z - 1} = z^6 + z^5 + z^4 + z^3 + z^2 + z + 1$ and

$\dfrac{z^7 - 1}{z - 1} = \left(z^2 - 2z\cos\frac{2\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{6\pi}{7} + 1\right)$

iii) Now

$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = \left(z^2 - 2z\cos\frac{2\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{6\pi}{7} + 1\right)$

Substituting $z = 1$: $7 = \left(1^2 - 2\cos\frac{2\pi}{7} + 1\right)\left(1^2 - 2\cos\frac{4\pi}{7} + 1\right)\left(1^2 - 2\cos\frac{6\pi}{7} + 1\right)$

$$7 = \left(1^2 - 2\cos\tfrac{2\pi}{7} + 1\right)\left(1^2 - 2\cos\tfrac{4\pi}{7} + 1\right)\left(1^2 - 2\cos\tfrac{6\pi}{7} + 1\right)$$
$$= \left(2 - 2\cos\tfrac{2\pi}{7}\right)\left(2 - 2\cos\tfrac{4\pi}{7}\right)\left(2 - 2\cos\tfrac{6\pi}{7}\right)$$
$$= 8\left(1 - \cos\tfrac{2\pi}{7}\right)\left(1 - \cos\tfrac{4\pi}{7}\right)\left(1 - \cos\tfrac{6\pi}{7}\right)$$

i.e. $\left(1 - \cos\tfrac{2\pi}{7}\right)\left(1 - \cos\tfrac{4\pi}{7}\right)\left(1 - \cos\tfrac{6\pi}{7}\right) = \tfrac{7}{8}$ as required

b)

S	U	R	E	N	D
		R	E		
		R			

i) Place one letter down as a marker. Then there are 8 remaining letters, 3 of one kind and 2 of another.

Number of arrangements $= \dfrac{8!}{3!2!} = 3360$

ii) Tie 'NERDS' together as one group and place this group as the marker. Then there are 4 remaining letters including a pair of R's

Number of arrangements $= \dfrac{4!}{2!} = 12$

$\therefore$ $P(\text{seeing NERDS}) = \dfrac{12}{3360} = \dfrac{1}{280}$

iii) $P(\text{NERDS in order}) = \dfrac{1}{9} \times \dfrac{2}{8} \times \dfrac{3}{7} \times \dfrac{1}{6} \times \dfrac{1}{5}$

But there are 5! Ways of ordering the letters of NERDS

$$\therefore \quad P\left(\text{NERDS in any order}\right) = \frac{1}{9} \times \frac{2}{8} \times \frac{3}{7} \times \frac{1}{6} \times \frac{1}{5} \times 5!$$
$$= \frac{2 \times 3 \times 5!}{9 \times 8 \times 7 \times 6 \times 5}$$
$$= \frac{1}{21}$$

End of Solutions